

# Super edge-graceful paths

Sylwia Cichacz<sup>\*†</sup>

AGH University of Science and Technology

and

University of Minnesota Duluth

Dalibor Froncek

University of Minnesota Duluth

Wenjie Xu

University of Minnesota Duluth

April 23, 2008

## Abstract

A graph  $G(V, E)$  of order  $|V| = p$  and size  $|E| = q$  is called super edge-graceful if there is a bijection  $f$  from  $E$  to  $\{0, \pm 1, \pm 2, \dots, \pm \frac{q-1}{2}\}$  when  $q$  is odd and from  $E$  to  $\{\pm 1, \pm 2, \dots, \pm \frac{q}{2}\}$  when  $q$  is even such that the induced vertex labeling  $f^*$  defined by  $f^*(x) = \sum_{xy \in E(G)} f(xy)$  over all edges  $xy$  is a bijection from  $V$  to  $\{0, \pm 1, \pm 2, \dots, \pm \frac{p-1}{2}\}$  when  $p$  is odd and from  $V$  to  $\{\pm 1, \pm 2, \dots, \pm \frac{p}{2}\}$  when  $p$  is even.

We prove that all paths  $P_n$  except  $P_2$  and  $P_4$  are super edge-graceful.

## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. We use standard terminology and notation of graph theory.

---

<sup>\*</sup>The work was supported by Fulbright Scholarship nr 15072441.

<sup>†</sup>cichacz@agh.edu.pl

Let  $G = (V, E)$  be a graph with  $p$  vertices and  $q$  edges. A vertex labeling of a graph  $G$  is a function from  $V(G)$  into  $\mathbb{N}$ . A. Rosa [7] introduced the graceful graph labeling. A graph  $G$  is *graceful* if there exists an injection from the vertices of  $G$  to the set  $\{0, \dots, q\}$  such that, when each edge  $xy$  is assigned the label  $|f(x) - f(y)|$ , the resulting edge labels are distinct.

The edge-graceful labeling was introduced by S.P. Lo [6]. A graph  $G$  is *edge-graceful* if the edges can be labeled by  $1, 2, \dots, q$  such that the vertex sums are distinct (mod  $p$ ). A necessary condition for a graph with  $p$  vertices and  $q$  edges to be edge-graceful is that  $q(q+1) \equiv \frac{p(p-1)}{2} \pmod{p}$ .

J. Mitchem and A. Simoson [2] defined super edge-graceful labeling which is a stronger concept than edge-graceful for some classes of graphs. Define an edge labeling as a bijection

$$f : E(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm \frac{q-1}{2}\} \text{ for } q \text{ odd}$$

or

$$f : E(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm \frac{q}{2}\} \text{ for } q \text{ even.}$$

For every vertex  $x \in V(G)$ , define the induced vertex labeling of  $x$  as  $f^*(x) = \sum_{xy \in E(G)} f(xy)$ . If  $f^*$  is a bijection

$$f^* : V(G) \rightarrow \{0, \pm 1, \pm 2, \dots, \pm \frac{p-1}{2}\} \text{ for } p \text{ odd}$$

or

$$f^* : V(G) \rightarrow \{\pm 1, \pm 2, \dots, \pm p\} \text{ for } p \text{ even,}$$

then the labeling  $f$  is *super edge-graceful*.

S.-M. Lee and Y.-S. Ho showed that all trees of odd order with three even vertices are super edge-graceful [4]. In [3] P.-T. Chung, S.-M. Lee, W.-Y. Gao, and K. Schaffer asked which paths are super edge-graceful. In this paper we show that all paths  $P_n$  except  $P_2$  and  $P_4$  are super edge-graceful.

## 2 Super edge-gracefulness of $P_n$

Let  $P_m = x_1, x_2, \dots, x_m$  be a path with an edge labeling  $f$  and  $P'_m = x'_1, x'_2, \dots, x'_m$  be a path with an edge labeling  $f'$ . If for every  $i$ ,  $1 \leq i \leq m-1$  we have  $f(x_i x_{i+1}) = -f'(x_i x_{i+1})$  then the labeling  $f$  is called *inverse* of  $f'$ .

**Theorem 1** *The path  $P_n$  is super edge-graceful unless  $n = 2, 4$ .*

*Proof.* It is obvious that  $P_2$  is not super edge-graceful.  $P_4$  is not super edge-graceful since the edge label set is  $\{0, -1, 1\}$  and the vertex set is  $\{-2, -1, 1, 2\}$ , but no two edge labels will sum up to 2 or  $-2$ . A labeling of  $P_3$  is trivial. We label the edges along  $P_6$  by  $(1, 2, 0, -2, -1)$ , whereas along  $P_{10}$  by  $(4, 1, -4, 0, 3, -1, 2, -3, -2)$  (see Figure 1).

Assume from now on that  $n \geq 5$  and  $n \neq 6, 10$ . The basic idea of our proof is to consider a path  $P_n$  as a union of paths  $Q_1$  and  $Q_2$  joined by an edge with label 0. We consider the following cases (for the sake of completeness we will include cases for odd paths  $P_n$  for  $n \geq 5$ , which were proved in [4]):

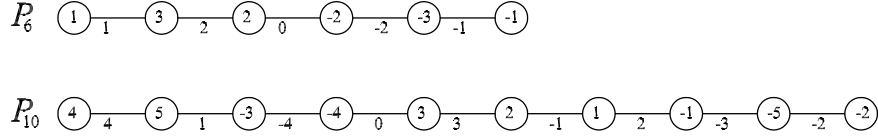


Figure 1: A super edge-graceful labeling  $P_6$  and  $P_{10}$ .

*Case 1.*  $n \equiv 1(\text{mod } 4)$ .

It follows that  $n = 4k + 1$  where  $k$  is a positive integer. Then we can find a super edge-graceful labeling as follows:

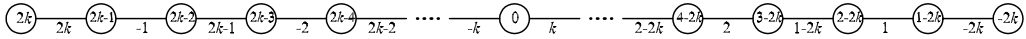


Figure 2: A super edge-graceful labeling  $P_n$  for  $n \equiv 1(\text{mod } 4)$ .

Notice that  $P_{4k+1}$  consists of two paths  $P_{2k+1}$  and  $P'_{2k+1}$  with edge labelings  $f$  and  $f'$ , respectively, such that  $f$  is inverse of  $f'$ .

*Case 2.*  $n \equiv 3(\text{mod } 4)$ .

It follows that  $n = 4k + 3$  where  $k$  is a positive integer. Similarly as in the previous case we find a super edge-graceful labeling. Notice that  $P_{4k+3}$  consists of two paths  $P_{2k+2}$  and  $P'_{2k+2}$  with edge labelings  $f$  and  $f'$ ,

respectively, such that  $f$  is inverse of  $f'$ .

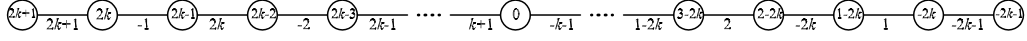


Figure 3: A super edge-graceful labeling  $P_n$  for  $n \equiv 3(\text{mod } 4)$ .

*Case 3.*  $n \equiv 0(\text{mod } 8)$ .

It follows that  $n = 8k$  for some positive integer  $k$ . We will consider  $P_n$  as a union of  $P_{4k}$  and  $P'_{4k}$  with edge labelings  $f$  and  $f'$ , respectively, such that  $f$  is inverse of  $f'$ . We join the paths  $P_{4k}$  and  $P'_{4k}$  by an edge with label 0. We label the edges along  $P_{4k}$  by  $(4k-1, -1, 4k-2, -2, 4k-3, -3, \dots, -k+1, 3k, k, -3k+1, k+1, -3k+2, k+2, \dots, -2k-1, 2k-1, -2k)$  (see Figure 4).

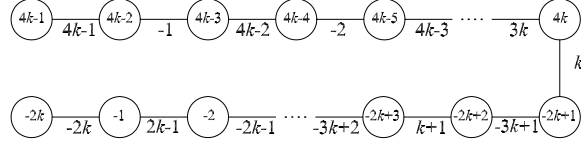


Figure 4: A labeling of  $P_{4k}$ .

*Case 4.*  $n \equiv 6(\text{mod } 8)$ .

Let  $n = 8k + 6$  for some positive integer  $k$ . As in Case 3 we will consider  $P_n$  as a union of  $P_{4k+3}$  and  $P'_{4k+3}$  with inverse labelings  $f$  and  $f'$ , respectively, joined by an edge with label 0. We label the edges along  $P_{4k+3}$  by  $(4k+2, -1, 4k+1, -2, 4k, -3, \dots, 3k+3, -k, 3k+2, k+1, -3k-1, k+2, -3k, k+3, \dots, 2k, -2k-2, 2k+1)$  (see Figure 5).

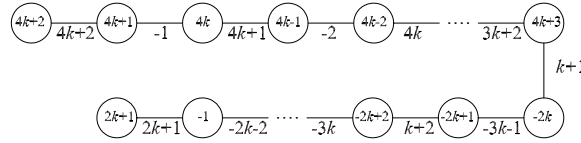


Figure 5: A labeling of  $P_{4k+3}$ .

*Case 5.*  $n \equiv 4(\text{mod } 8)$ .

Let  $n = 8k + 4$  for some positive integer  $k$ . We will consider  $P_n$  as a union

of paths  $Q_1$  and  $Q_2$  of lengths  $4k + 3$  and  $4k + 1$ , respectively, joined by an edge with label 0. We label the edges along  $Q_1$  by  $(2k + 1, -2k - 2, 2k, -2k - 3, \dots, k + 2, -3k - 1, k + 1, 3k + 1, -k - 1, 3k, \dots, -2k + 1, 2k + 2, -2k, -2k - 1)$  (see Figure 6).

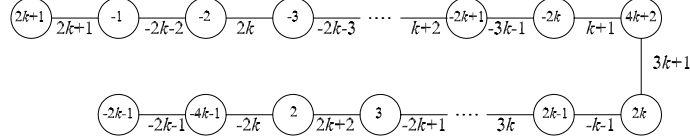


Figure 6: A labeling of  $Q_1$ .

Further, we label edges along  $Q_2$  by  $(4k + 1, -1, 4k, -2, \dots, -k + 1, 3k + 2, -k, -3k - 2, k, -3k - 3, \dots, -2k, 2, -4k - 1, 1)$  (see Figure 7).

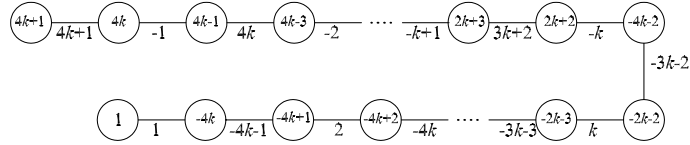


Figure 7: A labeling of  $Q_2$ .

*Case 6.*  $n \equiv 2 \pmod{16}$ .

Let  $n = 16k + 2$  for some positive integer  $k$ . As in Case 5 we consider  $P_n$  as a union of paths  $Q_3$  and  $Q_4$  of lengths  $8k + 2$  and  $8k$ , respectively, joined by an edge with label 0. We label the edges along  $Q_3$  by  $(4k, -4k - 1, 4k - 1, -4k - 2, \dots, -6k + 1, 2k + 1, -6k, 2k - 1, 6k, -2k + 1, 6k - 1, \dots, 4k + 2, -4k + 1, 4k + 1, -4k)$  (see Figure 8). Then we label edges of  $Q_4$  by  $(8k, -2, 8k -$

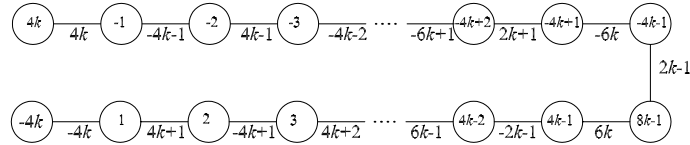


Figure 8: A labeling of  $Q_3$ .

$1, -3, \dots, 6k + 2, -2k, 6k + 1, 2k, -6k - 2, 2k - 3, -6k - 1, 2k - 2, -6k - 4, 2k - 5, -6k - 3, 2k - 4, -6k - 6, 2k - 7, -6k - 5, -2k + 6, -6k - 8, \dots, -8k + 6, 5, -8k + 7, 6, -8k + 4, 3, -8k + 5, 4, -8k + 2, 1, -8k + 3, 2, -8k, -1, -8k + 1)$  (see Figure 9).

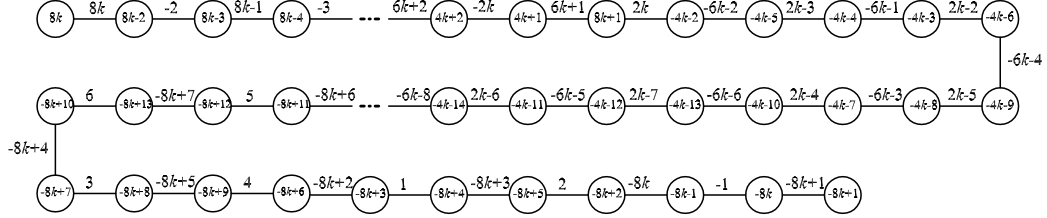


Figure 9: A labeling of  $Q_4$ .

*Case 7.  $n \equiv 10 \pmod{16}$ .*

Let  $n = 16k + 10$  for some positive integer  $k$ . As in previous cases we consider  $P_n$  as a union of paths  $Q_5$  and  $Q_6$  of lengths  $8k + 6$  and  $8k + 4$ , respectively, joined by an edge with label 0. We label the edges along  $Q_5$  by  $(4k + 2, -4k - 3, 4k + 1, -4k - 4, -6k - 2, 2k + 2, -6k - 3, 2k, 6k + 3, -2k + 2, 6k + 2, \dots, 4k + 4, -4k - 1, 4k + 3, -4k - 2)$  (see Figure 10).

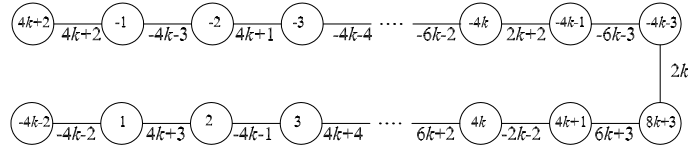


Figure 10: A labeling of  $Q_5$ .

Then we label  $Q_6$  by  $(8k + 4, -2, 8k + 3, -3, \dots, 6k + 5, -2k - 1, 6k + 4, 2k + 1, -6k - 5, 2k - 2, -6k - 4, 2k - 1, -6k - 7, 2k - 4, -6k - 6, 2k - 3, -6k - 9, 2k - 6, -6k - 8, 2k - 5, -6k - 11, \dots, 6, -8k + 4, 7, -8k + 1, 4, -8k + 2, 5, -8k - 12, -8k, 3, -8k + 3, 1, -8k - 2, -1, -8k - 4)$  (see Figure 11). ■

The corollary follows immediately from the proof of Theorem 1.

**Corollary 2** *If  $n$  is odd then the cycle  $C_n$  is super edge-graceful.*

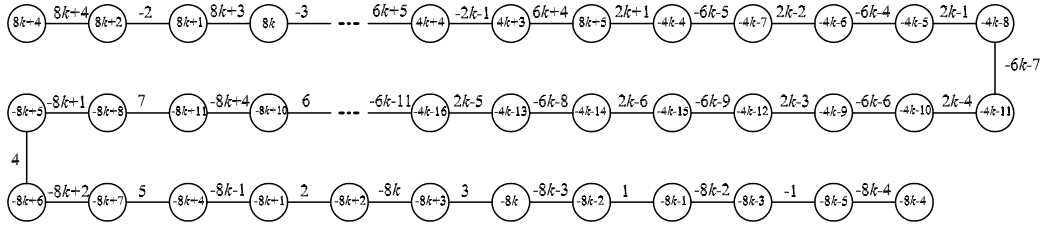


Figure 11: A labeling of  $Q_6$ .

*Proof.* Since  $n$  is odd then we can use the labeling for a path  $P_n$ , and after that, by joining together the end vertices of the path by the edge with label 0, we obtain a cycle  $C_n$  which is super edge-graceful. ■

## References

- [1] J.A. Gallian, *A Dynamic Survey of Graph Labeling*, The Electronic Journal of Combinatorics (2005) 20 Dec. 2006.
- [2] J. Mitchem and A. Simoson, *On edge-graceful and super edge-graceful labelings of graphs*, Ars Comb. **37** (1994) 97–111.
- [3] P.-T. Chung, S.-M. Lee, W.-Y. Gao, K. Schaffer, *On the super edge-graceful trees of even orders*, Congressus Numerantium **181** (2006) 5–17.
- [4] S.-M. Lee and Y.-S. Ho, *All trees of odd order with three even vertices are super edge-graceful*, J. Combin. Math. Combin. Comput. **62** (2007) 53–64.
- [5] S.-M. Lee, L. Wang and K. Nowak, *On the edge-graceful trees conjecture*, J. Combin. Math. Combin. Comput. **54** (2005) 83–98.
- [6] S.P. Lo, *On edge-graceful labelings of graphs*, Congressus Numerantium **50** (1985) 231–241.
- [7] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris. (1967) 349–355.